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Cerenkov Radiation in a Magneto-Ionic Medium (with Application to the Generation of Low-Frequency Electromagnetic Radiation in the Exosphere by the Passage of Charged Corpuscular Streams)

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ČERENKOV RADIATION IN A MAGNETO-IONIC MEDIUM (WITH APPLICATION TO THE GENERATION OF LOW-FREQUENCY ELECTROMAGNETIC RADIATION IN THE EXOSPHERE BY THE PASSAGE OF CHARGED CORPUSCULAR STREAMS)

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The electromagnetic field of a uniformly moving point charge in a magneto-ionic medium is evaluated by representing the field due to the moving charge in an angular spectrum of plane waves. A brief discussion of the frequency ranges satisfying the Čerenkov coherence condition is given. It is shown that Čerenkov radiation in the exosphere produces low-frequency hiss. The spectral and polar distributions of radiation over the low-frequency band were computed numerically for plasma and gyro frequencies occurring in the exosphere and for particle speeds of the order of solar corpuscular cloud speeds. An interesting feature of the radiation is that the more slowly the particle moves the greater is the emitted radiation.

List of symbols used in the text

c	free space speed of electromagnetic waves
$\boldsymbol{e}$	charge on particle
m	particle rest mass
v	particle speed
$\lvert\mathbf{B}_0\rvert$	magnitude of external magnetic field
ω	angular wave frequency
μ	refractive index
k	angular wave number
N	number density of electrons
μ_0	permeability of free space
ϵ_0	permittivity of free space
Z	$(\mu_0/\epsilon_0)^{\frac{1}{2}}$ vacuum impedance
θ	angle between wave normal and $\mathbf{B}_0$
(l, m, n)	direction cosines of wave normal

$p = (N\boldsymbol{e}^2/\epsilon_0 m)^{\frac{1}{2}}$ plasma frequency

$\Omega = \boldsymbol{e}B_0/m$ gyro frequency

$X = p^2/\omega^2$

$Y = \Omega/\omega$

$\epsilon = Y^2/(1-X)$

$$\alpha = \frac{1-X-Y^2}{(1-X)^2-Y^2}, \quad \beta = \frac{1}{1-X}, \quad \gamma = \frac{\mathrm{i}XY}{(1-X)^2-Y^2}$$

x, y, z	field co-ordinates
$\mathrm{d}W_i/\mathrm{d}\omega$	power per unit frequency per unit path length in ith mode.

1. Introduction

The fundamental theory of Čerenkov radiation produced by a point charge moving with uniform velocity through an isotropic (dielectric) medium was given by Frank & Tamm (1937) and Frank (1939). Since then many writers have considered the electrodynamics of uniformly moving point charges in various types of media. Angular distributions of radiation in such media as isotropic ferrodielectric and crystalline plates have been calculated by Pafomov (1961), and Muzicar (1961) has studied the Čerenkov effect in uniaxial crystals. Kolomenskii (1953) solved the Čerenkov problem in a gyrotropic medium, such as quartz, in considerable detail and also appears to have been the first to tackle the problem of satisfying the coherence condition and calculating the radiated power in an anisotropic medium Kolomenskii (1956).

A charge moving with speed v through a medium of refractive index μ will emit Čerenkov radiation at an angle α' with respect to its direction of motion if the condition for coherent radiation, namely

$$\cos\alpha' = c/\mu v \tag{1·1}$$

is satisfied. Thus Čerenkov radiation is emitted if the particle speed v is greater than the phase velocity of propagation of waves (c/μ) in the medium. In a magneto-ionic medium two distinct waves characterized by the Appleton–Hartree refractive indices μ_i $(i = 1, 2)$

may propagate. The speed of phase propagation of these waves, c/μ_i, is a function of the wave frequency ω, the plasma and gyro frequencies of the medium (p, Ω) and the angle θ between the wave normal and magnetic field. If ψ is the angle between the particle's track and the magnetic field the coherence condition for radiation in the ith mode becomes

$$\cos(\theta - \psi) = c/\mu_i(\theta)\, v. \tag{1·2}$$

For conditions existing in the earth's exosphere (Storey 1953), it is now well known that the velocity of phase propagation of low-frequency waves in the extraordinary mode may be very small compared to the vacuum velocity of light. The generation of radio waves through the Čerenkov mechanism by fast charged solar clouds passing through the exosphere is therefore always possible.

Ideally the problem would involve that of a finite bunch of charged particles spiralling in the earth's dipole field. The model considered in the text is that of a point charge moving with uniform velocity, v, from $-\infty$ to ∞ parallel to the line of action of the magnetic field in an infinite, homogeneous, collisionless magneto-ionic medium. The idealizations of the point charge and the homogeneity of the medium will be valid so long as (a) the wavelength of the emitted radiation is very much greater than the solar cloud's dimensions and that (b) the plasma and gyro frequencies change very little over distances of that order. The assumption of a collisionless medium will hold if the wave, gyro and plasma frequencies are all very much larger than the collision frequency of electrons with heavy particles.

Most writers evaluate the electromagnetic field produced by the uniformly moving point charge in terms of the scalar and vector potentials ϕ and $\mathbf{A}$. In the present paper the electromagnetic field due to the uniform motion of charged particle through a collisionless magneto-ionic medium, which is both dispersive and anisotropic, is written down as an angular spectrum of plane waves (Clemmow 1950). This method is particularly appropriate for fields produced by surface current distributions, and with this in mind the moving charge is represented as a current density in the x-y plane

$$K_x = ev\delta(x - vt)\,\delta(y), \quad K_y = 0$$

when the charge moves along the x axis parallel to the direction of the magnetostatic field.

In § 2 the refractive index, $\mu(l)$, of a plane wave propagating in a magneto-ionic medium is derived for the case of the magnetic field parallel to the x axis and the electromagnetic field caused by a surface current distribution in the x-y plane is written down as an angular spectrum of plane waves. The expressions for the field generated and radiation emitted by the uniform motion of a charged particle along the line of action of the magnetostatic field are given in § 3 employing the general theory outlined in the previous section. In § 4 the coherence condition and the Čerenkov frequency ranges are discussed briefly with particular reference to the nature of the low-frequency band. A discussion of the propagation of low-frequency Čerenkov radiation in the earth's exosphere is presented in § 5. It is shown that Čerenkov radiation due to streams of solar-charged particles will give rise to frequency versus time spectra which will be observed on earth as solid bands of hiss, the mean frequency of which descend slowly with time of arrival. The last section presents accurate approximations for the spectral and angular distributions of the radiation over the low-frequency bands and some numerically computed graphs of these quantities for specific values of the gyro and plasma frequencies at various distances from the earth's centre and for particle speeds appropriate to the estimated speed of solar clouds in the exosphere. An interesting feature of the radiation is that the radiation emitted per unit path length increases as the particle velocity decreases. The radiation generated by a particle moving in a plasma immersed in an infinite magnetic field is discussed in the appendix and there is obtained an estimate of energy variation with time for a particle losing energy by Čerenkov emission.

2. The plane wave spectrum in a magneto-ionic medium

(a) *The refractive index of a plane wave propagating in a magneto-ionic medium*

With the time factor $e^{i\omega t}$ suppressed Maxwell's equations in a magneto-ionic medium are

$$\operatorname{curl}\mathbf{E} = -i\omega\mu_0\mathbf{H}, \tag{2·1}$$

$$\operatorname{curl}\mathbf{H} = i\omega\epsilon_0\mathbf{KE}, \tag{2·2}$$

in which $\mathbf{K}$ is the familiar magneto-ionic tensor which characterizes the anisotropy of the medium, and ϵ_0 and μ_0 are the vacuum permittivity and permeability. Since $\operatorname{div}\mathbf{H} = 0$ we take a plane wave of the form

$$\mathbf{H} = \left(P, Q\frac{lp+mQ}{n}\right)e^{ik(lx+my-nz)}, \tag{2·3}$$

where k is the vacuum wave number ω/c, and $(l, m, -n)$ is the direction of propagation of the wave normal. For an isotropic medium n is uniquely determined, apart from sign, in terms of l and m and P and Q can both be chosen arbitrarily. In an anisotropic medium there are in general four expressions for n in terms of l and m, and for each one a definite ratio of P to Q (a function of l and m). In a magneto-ionic medium there is considerable symmetry with respect to the direction of the magnetostatic field. If we take the plane $z = 0$ either normal to or containing the direction of the magnetic field, the four expressions for n in terms of l and m reduce to two, each with alternative signs. The refractive index μ of the plane wave, a function of l and m, is given by

$$\mu^2 = l^2+m^2+n^2. \tag{2·4}$$

Here we make use of the symmetry about the magnetic field by taking the plane $z = 0$ to contain its direction and further suppose it to be directed along the x axis.

From the expression for $\mathbf{H}$, equation (2·3), we find the corresponding components of the electric vector

$$\mathbf{E} = Z\mathbf{K}^{-1}\left\{\frac{lmP+(\mu^2-l^2)Q}{n}, \quad -\frac{(\mu^2-m^2)+lmQ}{n}, \quad -(mP-lQ)\right\}e^{ik(lx+my-nz)}, \tag{2·5}$$

where $Z = \surd(\mu_0/\epsilon_0)$, is the vacuum impedance and $\mathbf{K}^{-1}$ is the inverse of the tensor $\mathbf{K}$. When the external magnetic field, $\mathbf{B}_0$, acts along the x axis it is well known (Ratcliffe 1959) that the magneto-ionic tensor, in the absence of collisions, becomes

$$\mathbf{K} = \begin{pmatrix} 1-X & 0 & 0 \\ 0 & \dfrac{1-X-Y^2}{1-Y^2} & \dfrac{-iXY}{1-Y^2} \\ 0 & \dfrac{iXY}{1-Y^2} & \dfrac{1-X-Y^2}{1-Y^2} \end{pmatrix}, \tag{2·6}$$

with the inverse tensor $\mathbf{K}^{-1}$ given by

$$\mathbf{K}^{-1} = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \alpha & \gamma \\ 0 & -\gamma & \alpha \end{pmatrix}, \tag{2·7}$$

where for the sake of brevity

$$\beta = \frac{1}{1-X}, \quad \gamma = \frac{\mathrm{i}XY}{(1-X)^2-Y^2}, \quad \alpha = \frac{1-X-Y^2}{(1-X)^2-Y^2}. \tag{2·8}$$

Substitution of $\mathbf{K}^{-1}$ from equation (2·7) into equation (2·5) yields the following for the components of the electric field

$$\mathbf{E} = Z\Big\{\beta\frac{[lmP+(\mu^2-l^2)\,Q]}{n}, \quad -\Big[\frac{\alpha}{n}(\mu^2-m^2)+\gamma m\Big]P-l\Big(\frac{\alpha m}{n}-\gamma\Big)\,Q,$$
$$\Big[\frac{\gamma}{n}(\mu^2-m^2)-\alpha m\Big]P+l\Big(\frac{\gamma m}{n}+\alpha\Big)Q\Big\}\,\mathrm{e}^{\mathrm{i}k(lx+my-nz)}. \tag{2·9}$$

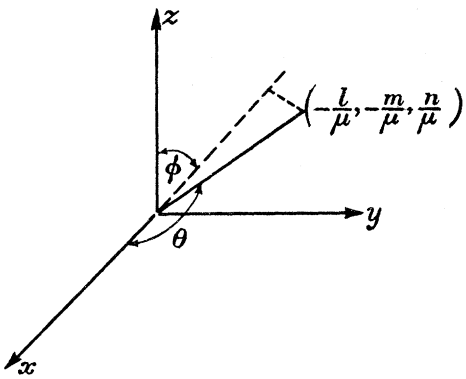

FIGURE 1. The direction cosines of the plane wave $\mathrm{e}^{\mathrm{i}k(lx+my-nz)}$.

The two ratios P/Q are now obtained by substituting equations (2·3) and (2·9) into equation (2·1). The equality of these two ratios will give the two expressions for the refractive indices, μ_i^2, as a function of l and m and then of course the ratios will be known for each μ_i^2. Equating the ratios P/Q obtained from the x and y components of equation (2·1) (the third ratio is redundant since $\operatorname{div}\mathbf{H} = 0$ is already guaranteed) gives the following formula for the refractive index

$$\mu_i^2 = 1 - \frac{X}{1-\frac{1}{2}\epsilon(1-l^2) \pm \sqrt{\{\frac{1}{4}\epsilon^2(1-l^2)^2+\epsilon l^2\}}}, \tag{2·10}$$

with $i = 1$ for the positive sign in the square root and $i = 2$ for the negative sign.

The ratio $(P/Q)_i$ is then given by (x component of equation (2·1))

$$\Big(1+\gamma\frac{l^2m}{n_i}-\alpha\mu_i^2\Big)\,P_i+\frac{\gamma l}{n_i}(\mu_i^2-l^2)\,Q_i = 0. \tag{2·11}$$

It can be seen from figure 1 that

$$-\frac{l}{\mu} = \cos\theta, \quad -\frac{m}{\mu} = \sin\theta\,\sin\phi, \quad \frac{n}{\mu} = \sin\theta\,\cos\phi. \tag{2·12}$$

The functional dependence of μ^2 on l, given by equation (2·10), is closely related to the well known Appleton–Hartree formula for the refractive index. In fact putting $l^2 = \mu^2\cos^2\theta$ into (2·10) and solving for μ^2 as a function of θ yields the Appleton–Hartree formula.

(b) *The angular spectrum of plane waves in a magneto-ionic medium*

The angular spectrum expansion of a Fourier frequency component of an electromagnetic field in a magneto-ionic medium may be written

$$\mathbf{H} = \sum_{i=1}^{2} \iint_{-\infty}^{\infty} \left\{ \pm P_i,\ \pm Q_i,\ \frac{lP_i + mQ_i}{n_i} \right\} e^{ik(lx+my\mp n_i z)}\, dl\, dm, \tag{2·13}$$

$$\mathbf{E} = Z \sum_{i=1}^{2} \iint_{-\infty}^{\infty} \left\{ \frac{\beta}{n_i}[lm P_i + (\mu_i^2 - l^2) Q_i],\ -\left[\frac{\alpha}{n_i}(\mu_i^2 - m^2) + \gamma m\right] P_i - l\left(\frac{\alpha m}{n_i} - \gamma\right) Q_i, \right.$$
$$\left. \pm \left[\frac{\gamma}{n_i}(\mu_i^2 - m^2) - \alpha m\right] P_i \pm l\left(\frac{\gamma m}{n_i} + \alpha\right) Q_i \right\} e^{ik(lx+my\mp n_i z)}\, dl\, dm, \tag{2·14}$$

where $n_i = \sqrt{(\mu_i^2 - l^2 - m^2)}$, with the upper sign referring to propagation into the region $z > 0$, the lower for $z < 0$.

If the electromagnetic field is produced by a plane surface current distribution in the x-y plane with Fourier time components

$$J = (J_x, J_y, 0), \tag{2·15}$$

then the discontinuity of $\mathbf{H}$ across $z = 0$ is matched to the current as follows:

$$\left.\begin{aligned} J_x &= H_y(z=0-) - H_y(z=0+) = 2H_y(z=0-), \\ J_y &= H_x(z=0+) - H_x(z=0-) = 2H_x(z=0+), \end{aligned}\right\} \tag{2·16}$$

whence, from equations (2·13),

$$\left.\begin{aligned} J_x &= -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (Q_1 + Q_2)\, e^{ik(lx'+my')}\, dl\, dm, \\ J_y &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (P_1 + P_2)\, e^{ik(lx'+my')}\, dl\, dm, \end{aligned}\right\} \tag{2·17}$$

with the inverse transforms,

$$\left.\begin{aligned} P_1 + P_2 &= \frac{k^2}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_y(x', y')\, e^{-ik(lx'+my')}\, dx'\, dy' = j_y, \\ Q_1 + Q_2 &= \frac{-k^2}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_x(x', y')\, e^{-ik(lx'+my')}\, dx'\, dy' = -j_x. \end{aligned}\right\} \tag{2·18}$$

The scheme is completed by utilizing equation (2·11) which establishes the ratio $(P/Q)_i = -a_i$. The solution of these equations for the spectrum function P_i, Q_i is

$$P_i = \frac{a_i}{(a_j - a_i)}(a_j j_x - j_y), \quad Q_i = \frac{-P_i}{a_i} \quad (i = 1, 2;\ j = 2, 1). \tag{2·19}$$

Substituting these expressions (equations (2·19)) into the equations for the frequency component of the field (equations (2·13) and (2·14)) and simplifying the coefficients of the current transforms, we find that the ith mode may be written

$$H_{x,i}^{(\omega)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{-\gamma\beta l(\mu_i^2 - l^2) j_x + [\mp n_i(\alpha - (\alpha^2 + \gamma^2)\mu_i^2) + (\mu_i^2 - l^2)$$
$$\times (\mp(\alpha - \beta) n_i - \gamma\beta m) + \gamma m] j_y\} \frac{e^{ik(lx+my\mp n_i z)}}{\alpha\beta n_i(\mu_i^2 - \mu_j^2)}\, dl\, dm,$$

$$H_{y,i}^{(\omega)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{-\beta[\mp n_i(1 - \alpha\mu_i^2) - \gamma m l^2] j_x - l[\mp(\alpha - \beta) m n_i + \gamma(1 - \beta m^2)] j_y\}$$
$$\times \frac{e^{ik(lx+my\mp n_i z)}}{\alpha\beta n_i(\mu_i^2 - \mu_j^2)}\, dl\, dm, \tag{2·20}$$

$$H^{(\omega)}_{z,i} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\beta[m(1-\alpha\mu_i^2)\mp\gamma n_i l^2] - l[\alpha-(\alpha^2+\gamma^2)\,\mu_i^2+(\alpha-\beta)\,n_i^2\pm\gamma\beta m n_i]\,j_y\} \times \frac{e^{ik(lx+my\mp n_i z)}}{\alpha\beta n_i(\mu_i^2-\mu_j^2)}\,dl\,dm,$$

$$E^{(\omega)}_{x,i} = \mathbf{Z}\beta\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\beta(1-\alpha\mu_i^2)\,(\mu_i^2-l^2)\,j_x - l[(\alpha m\pm\gamma n_i)-(\alpha^2+\gamma^2)\,m\mu_i^2]\,j_y\} \times \frac{e^{ik(lx+my\mp n_i z)}}{\alpha\beta(\mu_i^2-\mu_j^2)\,n_i}\,dl\,dm,$$

$$E^{(\omega)}_{y,i} = \mathbf{Z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{-\beta l[(\mp\gamma n_i-\alpha m)-(\alpha^2+\gamma^2)\,m\mu_i^2]\,j_x + [(\alpha-(\alpha^2+\gamma^2)\,\mu_i^2)\,(1-\beta m^2)+(\alpha-\beta)\,n_i^2]\,j_y\} \frac{e^{ik(lx+my\mp n_i z)}}{\alpha\beta(\mu_i^2-\mu_j^2)\,n_i}\,dl\,dm,$$

$$E^{(\omega)}_{z,i} = \mathbf{Z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{-\beta l[(\mp\alpha n_i-\gamma m)\pm(\alpha^2+\gamma^2)\,n_i\mu_i^2]\,j_x - [\mp(\alpha^2+\gamma^2)\,(1-\beta\mu_i^2)\,m n_i+\gamma(1-\beta(\mu_i^2-l^2))]\,j_y\} \frac{e^{ik(lx+my\mp n_i z)}}{\alpha\beta(\mu_i^2-\mu_j^2)\,n_i}\,dl\,dm.$$

3. The electromagnetic field of a point charge moving uniformly along the magnetic field direction (Čerenkov radiation)

We now consider the electromagnetic field generated by the uniform motion of a particle, charge $\boldsymbol{e}$, directed along the line of action of the magnetostatic field (the x axis). The charge is assumed to move with uniform velocity v from $x = -\infty$ to $x = \infty$ passing through the origin at time $t = 0$ and is represented as a plane surface current density in the x-y plane with components

$$K_x = \boldsymbol{e}v\delta(x'-vt)\,\delta(y'), \quad K_y = 0, \tag{3·1}$$

with the inverse Fourier time transform

$$J_x = (\boldsymbol{e}/2\pi)\,\delta(y')\,e^{-i\omega x/v}, \quad J_y = 0 \tag{3·2}$$

so that the space time Fourier transform defined by equations (2·18) is

$$j_x = -(\boldsymbol{e}/16\pi^2)\,\delta\{l+(c/v)\}, \quad j_y = 0. \tag{3·3}$$

The delta function appearing in the Fourier space time transform of the current indicates that the field only exists in the direction $l = -c/v$ which is of course the Čerenkov coherence condition for radiation,

$$\cos\theta = c/\mu_i(\theta)\,v. \tag{3·4}$$

The frequency components of the ith mode of the electromagnetic field due to the moving charge are found by simply inserting the current transform, defined by equation (3·3), into the angular spectrum expansion, equations (2·20). The presence of the delta function makes the l integration trivial while the integration over m can be carried out by using the Sommerfeld representation of the zero-order Hankel function $H_0^{(2)}$,

$$\int_{-\infty}^{\infty} \frac{\exp\{ik(my\mp\surd(m_{0_i}^2-m^2)\,z)\}}{(\surd m_{0_i}^2-m^2)}\,dm = \pi H_0^{(2)}(k_i\rho), \tag{3·5}$$

$$m_{0_i} = (\mu_i^2-c^2/v^2)^{\frac{1}{2}}, \quad k_i = k m_{0_i}, \quad y^2+z^2 = \rho^2, \quad \tan\phi = y/z. \tag{3·6}$$

In cylindrical co-ordinates the field, in terms of $H_0^{(2)}(k_i\rho)$ and $H_0^{(2)\prime}(k_i\rho)$, becomes

$$\begin{aligned} E_{x,i}^{(\omega)} &= \frac{e\mu_0}{8\pi}\omega\beta\frac{(\alpha^{-1}-\mu_i^2)\,(\mu_i^2-c^2/v^2)}{\mu_i^2-\mu_j^2}\,\mathrm{e}^{-\mathrm{i}\omega x/v}H_0^{(2)}(k_i\rho),\\ E_{\rho,i}^{(\omega)} &= \frac{-\mathrm{i}e\mu_0}{8\pi}\omega\frac{c}{v}\frac{(\mu_i^2-c^2/v^2)^{\frac{1}{2}}\,(1-(\alpha+\gamma^2/\alpha)\,\mu_i^2)}{\mu_i^2-\mu_j^2}\,\mathrm{e}^{-\mathrm{i}\omega x/v}H_0^{(2)\prime}(k_i\rho),\\ E_{\phi,i}^{(\omega)} &= \frac{\mathrm{i}e\mu_0}{8\pi}\omega\frac{c}{v}\frac{(\mu_i^2-c^2/v^2)^{\frac{1}{2}}}{\mu_i^2-\mu_j^2}\frac{\gamma}{\alpha}\,\mathrm{e}^{-\mathrm{i}\omega x/v}H_0^{(2)\prime}(k_i\rho), \end{aligned} \tag{3·7}$$

$$\begin{aligned} H_{x,i}^{(\omega)} &= \frac{e}{8\pi}\frac{\omega}{v}\frac{(\mu_i^2-c^2/v^2)}{\mu_i^2-\mu_j^2}\frac{\gamma}{\beta}\,\mathrm{e}^{-\mathrm{i}\omega x/v}H_0^{(2)}(k_i\rho),\\ H_{\rho,i}^{(\omega)} &= \frac{-\mathrm{i}e}{8\pi}\frac{\omega c}{v^2}\frac{(\mu_i^2-c^2/v^2)^{\frac{1}{2}}}{\mu_i^2-\mu_j^2}\frac{\gamma}{\alpha}\,\mathrm{e}^{-\mathrm{i}\omega x/v}H_0^{(2)\prime}(k_i\rho),\\ H_{\phi,i}^{(\omega)} &= \frac{-\mathrm{i}e}{8\pi}\frac{\omega}{c}\frac{(\mu_i^2-c^2/v^2)^{\frac{1}{2}}\,(\alpha^{-1}-\mu_i^2)}{\mu_i^2-\mu_j^2}\,\mathrm{e}^{-\mathrm{i}\omega x/v}H_0^{(2)\prime}(k_i\rho) \quad (i=1,2;\, j=2,1). \end{aligned} \tag{3·8}$$

The refractive index, μ_i^2, given by equation (2·10) is evaluated at $l^2 = c^2/v^2$. When $c^2/v^2 > \mu_i^2$ we choose the branch of $(\mu_i^2-c^2/v^2)^{\frac{1}{2}}$ to be $-\mathrm{i}(c^2/v^2-\mu_i^2)^{\frac{1}{2}}$. The field generated by the moving charge is radiating or evanescent according as the particle speed, v, is greater or less than the wave speed c/μ_i. The total Čerenkov radiation emitted by the charge per unit length of its path is easily evaluated using either the Poynting theorem or finding the work done by the electric field on the particle. Omitting the details the power spectrum of the ith mode is

$$\frac{\mathrm{d}W_i}{\mathrm{d}\omega} = \frac{e^2\mu}{4\pi}\omega\beta\frac{(\mu_i-\alpha^{-1})\,(\mu_i^2-c^2/v^2)}{\mu_i^2-\mu_j^2} \quad (i=1,2;\, j=2,1) \tag{3·9}$$

over frequency ranges satisfying $c^2/\mu_i^2v^2 < 1$. The formula for the power spectrum in an isotropic medium is recovered by making the electron number density vanishingly small, $X \to 0$, associating c with the speed of electromagnetic waves in the medium and noting that

$$\frac{\alpha\mu_i^2-1}{\mu_i^2-\mu_j^2} = 1 \mp \tfrac{1}{2}Y^2\left(\frac{c^2}{v^2}-1\right)\bigg/\left\{\tfrac{1}{2}Y^2\left(\frac{c^2}{v^2}-1\right)^2+Y^2\frac{c^2}{v^2}\right\}^{\frac{1}{2}}, \tag{3·10}$$

(with $i=1, j=2$ for the upper sign and $i=2, j=1$ for the lower), the addition of both modes yields the well-known formula for the Čerenkov power spectrum

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{e^2\mu_0}{4\pi}\omega\left(1-\frac{c^2}{v^2}\right) \quad \left(\frac{v}{c}>1\right). \tag{3·11}$$

The formula for the polar diagram of Čerenkov radiation is readily obtained from equations (3·9) which may be written

$$\mathrm{d}W_i = P_i\,\mathrm{d}\omega, \quad P_i = \mathrm{d}W_i/\mathrm{d}\omega. \tag{3·12}$$

The coherence condition, equation (3·4), establishes the relation between θ, the wave normal and ω the wave frequency so that

$$\cos^2\theta_i = g_i(\omega, p, \Omega, c/v), \tag{3·13}$$

where the function g_i is given in the next section. The polar diagram of the radiation is (from equations (3·12) and (3·13))

$$\mathrm{d}W_{i,\,\mathrm{ang.}} = P_i\cos\theta_i(\omega)/\pi g_i'(\omega). \tag{3·14}$$

4. The Čerenkov coherence condition in a magneto-ionic medium: Čerenkov frequency ranges

The refractive index, μ, for the propagation of waves of angular frequency, ω, making an angle θ with the magnetic field direction in a collisionless magneto-ionic medium is given by the Appleton–Hartree formula

$$\mu^2 = 1 - \frac{X}{1-\frac{1}{2}\epsilon\sin^2\theta \pm \{(\frac{1}{2}\epsilon\sin^2\theta)^2 + Y^2\cos^2\theta\}^{\frac{1}{2}}}. \tag{4·1}$$

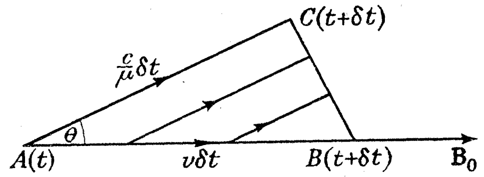

Figure 2. Geometrical illustration of the coherence condition.

A charged particle moving uniformly with speed v along the line of action of the magnetic field emits Čerenkov radiation at an angle θ with respect to its track (the field direction) if the coherence condition for radiation (see figure 2)

$$\cos^2\theta = c^2/\mu^2(\theta)\, v^2, \tag{4·2}$$

can be satisfied over certain frequency bands. Solving equations (4·2) for $\cos\theta^2$ we have

$$\cos^2\theta = \frac{\alpha^2}{2X}\frac{\left\{2\left(\frac{1}{X}-1\right)^2 - \frac{Y^2}{X}\left(\frac{2}{X^2}+\alpha^2-1\right) \pm \frac{Y}{X^{\frac{1}{2}}}\left[4\left(\frac{1}{X^2}-1\right)\alpha^2 - 1)^2\right]^{\frac{1}{2}}\right\}}{\left(\frac{1}{X}-1\right)^3 - \frac{Y^2}{X^2}\left(\frac{1}{X}+\alpha^2-1\right)} = g(\omega, p, \Omega, c/v), \tag{4·3}$$

where $\alpha = c/v$ where no confusion with the magneto-ionic parameter, α, defined by equation (2·8) is likely to arise.

Precisely the same equation for $\cos^2\theta$ is of course obtained by setting the argument of the delta function, appearing in the spectrum functions given by equations (3·3), to zero. The positive and negative signs in the radical of the Appleton–Hartree formula are associated with the ordinary and extraordinary modes, respectively. Perhaps it is worth observing that the positive and negative signs in the radical of equation (2·10) for the refractive index as a function of l $(=\mu\cos\theta)$ do not in general correspond to the ordinary and extraordinary waves. For example, the curve in figure 3 belongs to the extraordinary branch of the Appleton–Hartree refractive index and as the diagram illustrates both the branches μ_i $(i = 1, 2)$ correspond to the extraordinary mode. (The ordinary refractive index is a pure imaginary in this situation.)

The location of frequency ranges which satisfy the coherence condition may be found in two different ways. One method is to draw the vertical line $\mu\cos\theta = c/v$ on the various types of (μ, θ) curves which arise in a magneto-ionic medium and by finding the points of intersection between the line and the curve the Čerenkov frequency ranges may be established (see figure 4). Clemmow & Mullaly (1954) have examined the (μ, θ) curves of magneto-ionic theory, plotting them in Cartesian form and classifying them into eight distinct types by certain properties of their shape, and the regions of the (X, Y) plane to which these types correspond are specified analytically.

The type of curve shown in figure 4 corresponds to region 6*a* of the (X, Y) plane, whereas the curve in figure 3 belongs to region 6*b*. Region 6 refers to $Y > 1$ and $X > 1$. This region is subdivided into 6*a* and 6*b* by the curve $Y = 2(X-1)/(X-2)$, region 6*a* lying below this curve in the (X, Y) plane and 6*b* above it. Since the (μ, θ) curves belonging to region 6 are those appropriate to the propagation of low-frequency radio waves in the exosphere the properties of the (μ, θ) curves in other regions of the (X, Y) plane are not considered here.

The other method of locating the Čerenkov frequency ranges is to impose the radiation $0 \leqslant \cos^2\theta \leqslant 1$ on equation (4·3) and find what ranges of frequency satisfy this condition. The two methods are obviously equivalent. The present paper does not propose to carry

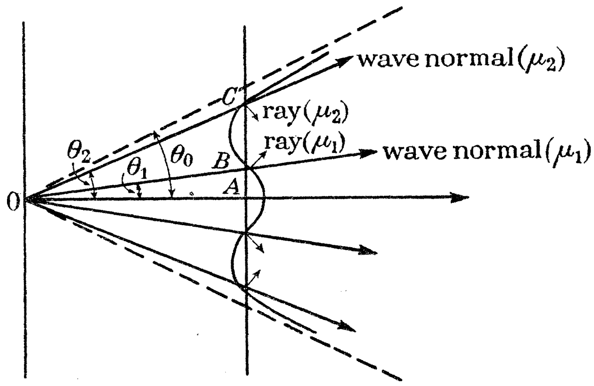

FIGURE 3. The (μ, θ) diagram appropriate to a Čerenkov frequency range in which there are two coherent directions θ_1 and θ_2 (region 6*b* of (X, Y) plane).

out a full examination of the coherence condition. The following brief discussion of equation (4·3) is adequate for our purpose. It is useful to state the asymptotic forms of the coherence condition.

(1) Very high frequencies: $X \ll 1$,

$$\cos^2\theta_i \sim c^2/v^2. \tag{4·4}$$

(2) Very low frequencies: $X \gg 1$,

$$\cos^2\theta_i \sim \frac{c^2}{2v^2X}\left\{\frac{Y^2}{X^2}\left(\frac{c^2}{v^2}-1\right) \pm \frac{Y}{X^2}\left[\frac{Y}{X^2}\left(\frac{c^2}{v^2}-1\right)^2 - 4\frac{c^2}{v^2}\right]^{\frac{1}{2}} - 2\right\}. \tag{4·5}$$

(3) Ultra relativistic particles: $v^2/c^2 \to 1$,

$$\cos^2\theta_i \sim \frac{1}{X}\left\{\frac{\left(\frac{1}{X}-1\right)^2 - \frac{Y^2}{X^2} \pm \frac{Y}{X^{\frac{1}{2}}}\left(\frac{1}{X}-1\right)^{\frac{1}{2}}}{\left(\frac{1}{X}-1\right)^3 - \frac{Y^2}{X^3}}\right\}. \tag{4·6}$$

(4) Slow particles: $v^2/c^2 \ll 1$,

$$\cos^2\theta_1 \sim c^2/v^2, \tag{4·7 a}$$

$$\cos^2\theta_2 \sim \frac{X+Y^2-1}{X^2+(c^2/v^2)\,Y^2}\left[1+\frac{Y^2}{X^2}\left(\frac{c^2}{v^2}+Y^2\right)\right]\frac{X}{Y^2}, \tag{4·7 b}$$

where $i = 1$ refers to negative sign in the radical and $i = 2$ for the positive sign.

Before proceeding to enumerate Čerenkov frequency bands it is natural to ask in what range must c/v lie to ensure $\cos^2\theta_i$ being real and positive. An upper limit to the velocity in terms of X and Y is obtained by the assertion that the quantity inside the square root in equation (4·3) must be positive and implies that

$$v \leqslant v_{\text{upp.}} = c/\alpha_1, \tag{4·8}$$

where
$$\alpha_1^2 = 1 + \frac{2(X-1)}{Y^2}\left\{1 \pm \left[\frac{X+Y^2-1}{\pm(X-1)}\right]^{\frac{1}{2}}\right\}, \tag{4·9}$$

It will be noted that if $X+Y^2 < 1$ (i.e. $\omega > (p^2+\Omega^2)^{\frac{1}{2}}$) no velocity exists satisfying the coherence condition. At first sight it would appear that a limiting minimum velocity below which there is no radiation would be found by stating $\cos^2\theta \geqslant 0$ which simplifies to the condition $c^2/v^2 \leqslant 0$ and means that all velocities are permissible. This ties in with the

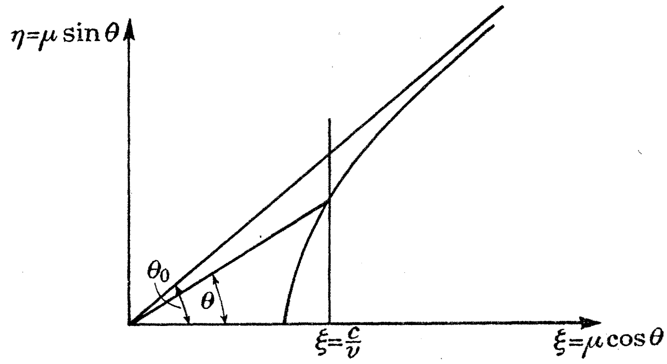

FIGURE 4. The (μ, θ) diagram appropriate to a Čerenkov frequency range in which there is only one coherent direction θ_2 (region 6a of (X, Y) plane).

existence of some branches of μ, the refractive index, reaching to infinity at certain angles of propagation to the field in some regions of the (X, Y) plane, which means that no matter how slow a particle moves there will always be a direction where the phase speed of an electromagnetic wave is slower. This property of the medium is discussed in the last section with reference to the increase of Čerenkov emission from a charge the more slowly it moves.

It can be shown that the frequency range $p \leqslant \omega \leqslant (p^2+\Omega^2)^{\frac{1}{2}}$ always satisfies the radiation condition $0 \leqslant \cos^2\theta_2 \leqslant 1$. In this range the other mode does not radiate since $\cos^2\theta_1 > 1$. At the plasma frequency, $X = 1$, the radiation is emitted along the direction of the charge's motion whereas at the upper limit $X+Y^2 = 1$ $(\omega = (p^2+\Omega^2)^{\frac{1}{2}})$ the emitted wave travels perpendicular to the particle direction. No radiation is generated with frequencies greater than $(p^2+\Omega^2)^{\frac{1}{2}}$. In any case waves with frequency greater than the plasma frequency will not penetrate the exosphere since they suffer reflexion. The low-frequency approximation to the coherence condition, equations (4·5), written in the form

$$\cos^2\theta_i = \omega^2/\omega_i^2, \tag{4·10}$$

$$\omega_i^2 = 2p^2\frac{v^2}{c^2}\left\{\frac{\Omega^2}{p^2}\left(\frac{c^2}{v^2}-1\right) \pm \frac{\Omega}{p}\sqrt{\left(\frac{\Omega^2}{p^2}\left(\frac{c^2}{v^2}-1\right)^2 - 4\frac{c^2}{v^2}\right)} - 2\right\}^{-1}, \tag{4·11}$$

(with $i = 1$ for positive radical, the negative for $i = 2$), shows that there is radiation in both modes in the ranges $0 \leqslant \omega \leqslant \omega_i$ and that $\omega_2 > \omega_1$. The extreme frequencies ω_i are real provided

$$\frac{\Omega^2}{p^2} \geqslant \frac{4c^2}{v^2}\left(\frac{c^2}{v^2}-1\right)^2. \tag{4·12}$$

This inequality is the low-frequency equivalent of the condition

$$\frac{4c^2}{v^2}\left(\frac{1}{X^2}-1\right)+\frac{Y^2}{X}\left(\frac{c^2}{v^2}-1\right)^2 \geqslant 0 \tag{4·13}$$

for $\cos^2\theta$ to be real. Highly accurate approximations to the upper limits ω_i of these low-frequency ranges can be found for $c^2/v^2 \gg 1$, $\Omega^2/p^2 \ll 1$, $(c^2/v^2)(\Omega^2/p^2) \gg 1$, and are given by

$$\omega_1 \simeq \frac{v^2p^2}{c^2\Omega}\left(1+\frac{v^2p^2}{c^2\Omega^2}\right), \tag{4·14}$$

$$\omega_2 \simeq \Omega\left(1-\frac{v^2p^2}{c^2\Omega^2}\right). \tag{4·15}$$

These approximations are valid over a large part of the earth's exosphere; for instance, for a particle speed of the order of estimated solar cloud speeds, i.e. roughly 1000 km/s, at 4 earth radii from the earth's centre where $\Omega^2/p^2 \sim 10^{-2}$, the approximations for ω_i certainly hold and show that waves associated with the refractive index μ_2 radiate over a much broader band than do 'μ_1' waves; the 'μ_2' wave radiating up to a frequency just short of the gyro frequency.

This simple analysis exhibits clearly the appearance of two low-frequency bands over which Čerenkov radiation is emitted. In the range $0 < \omega < \omega_1$ the (μ, θ) curves are those characteristic of region 6*b* of the (X, Y) plane (see figure 3) in which two waves of the same frequency may propagate at angles θ_1 and θ_2 to the magnetic field direction.

The remaining Čerenkov frequency range is $\omega_1 < \omega < \omega_2$, in which only one coherent direction, θ_2, given by equation (4·10), is possible. This of course agrees with the (μ, θ) curves having the properties specified by region 6*a* of the (X, Y) plane. The spectral and angular distribution of power in this range is given approximately by equations (6·1) and (6·2).

5. Propagation characteristics of low-frequency Čerenkov radiation in the exosphere

(a) *Whistler mode approximation*

A useful and accurate approximation to the refractive index for waves propagating in a magneto-ionic medium exists when the dimensionless parameters X and Y are such that $X \gg 1$, $X/Y^2 \gg 1$, $Y > 1$; namely, the well-known whistler mode refractive index for extraordinary waves given by the expression

$$\mu^2 \simeq 1+\frac{X}{Y\cos\theta-1} \sim \frac{X}{Y\cos\theta-1}. \tag{5·1}$$

The Čerenkov coherence condition, equations (4·2); may then be written

$$\frac{X\cos^2\theta}{Y\cos\theta-1} = \frac{c^2}{v^2} \tag{5·2}$$

which is quadratic in $\cos\theta$ the solutions agreeing with the approximations to $\cos^2\theta$ given by equations (4·10), (4·14) and (4·15). If the 1 in equation (5·1) had been retained a cubic equation in $\cos\theta$ would have been obtained, whereas the exact equation (4·2) yields a quadratic in $\cos^2\theta$, the solutions being equation (4·3), since the coefficient of $\cos^6\theta$ vanishes.

The condition for the existence of a coherent direction can be found by observing that the (μ, θ) curve cuts the ξ axis (i.e. $\theta = 0$) at

$$\mu_0^2 \simeq 1 + \frac{X}{Y-1}, \tag{5·3}$$

the minimum value of which occurs at $\omega = \frac{1}{2}\Omega$ and is

$$\mu_{0\,\text{min.}}^2 = 1 + 4p^2/\Omega^2, \tag{5·4}$$

Hence a coherent direction is possibly only if

$$c^2/v^2 \geqslant 1 + 4p^2/\Omega^2 \tag{5·5}$$

and is equivalent to inequality (4·12). All these results illustrate the validity of equation (5·1) for low-frequency Čerenkov radiation in the exosphere.

(b) *Group-ray refractive index; ray direction*

In a magneto-ionic medium, which is both dispersive and anisotropic, electromagnetic energy propagates at an angle, α, to the wave normal with speed U given by

$$U = c/\mu_{g-r}, \tag{5·6}$$

where
$$\mu_{g-r} = \mu' \cos\alpha, \quad \mu' = \mu + \frac{\mathrm{d}\mu}{\mathrm{d}\omega}, \quad \tan\alpha = -\frac{1}{2\mu^2}\frac{\partial}{\partial\theta}(\mu^2), \tag{5·7}$$

The symbols μ_{g-r} and μ' stand for the 'group-ray' and group refractive indices, respectively. The angle the ray makes with the magnetic field direction is $(\theta+\alpha)$. Evaluating these quantities using the whistler mode approximation, then inserting the Čerenkov relations between the angle of phase propagation, θ, and wave frequency, ω, given by equations (4·10) and (4·14) the following expressions are obtained:

$$\mu' \sim \frac{1}{\sqrt{2}}\frac{c^2}{v^2}\frac{\Omega^2}{p\omega}, \tag{5·8}$$

$$\tan\alpha \sim \frac{-1}{\sqrt{2}}\frac{c}{v}\frac{\Omega}{p}\tan\theta, \tag{5·9}$$

$$\mu_{g-r} \simeq \frac{c}{v}\frac{\Omega}{\omega}, \tag{5·10}$$

$$\theta+\alpha \sim \theta + \tan^{-1}\left(-\frac{1}{\sqrt{2}}\frac{c}{v}\frac{\Omega}{p}\tan\theta\right), \tag{5·11}$$

$$(\theta+\alpha)_{\text{max.}} \simeq \tan^{-1}(\sqrt{2}vp/c\Omega) + \tan^{-1}(-c\Omega/\sqrt{2}vp). \tag{5·12}$$

Noting that the upper limit of ω is $\Omega(1-v^2p^2/c^2\Omega^2)$ equations (5·6) and (5·10) show that the maximum group velocity is

$$U_{\text{max.}} \sim v(1-v^2p^2/c^2\Omega^2). \tag{5·13}$$

Therefore the electromagnetic energy always lags behind the particle as of course does the speed of phase propagation. It will be observed from equation (5·9) that at $\theta = 0$ the ray travels along the field direction. However, for relatively small changes, $\delta\theta$, of the wave normal direction the ray direction can change considerably since $c\Omega/\sqrt{2}\,vp \gg 1$. This rapid change of ray direction for small changes in wave normal direction can be clearly seen from the (μ, θ) curves belonging to region $6a$ of the (X, Y) plane when it is observed that the refractive index reaches asymptotically to infinity at an angle θ_0 given by

$$\tan^2\theta_0 = \frac{XY^2}{X+Y^2-1} - 1 \sim \frac{v^2p^2}{c^2\Omega^2} \ll 1, \tag{5·14}$$

which is an extremely small angle (see figure 5).

The maximum angle of deviation of the ray from the magnetic field direction is given by equation (5·12), which approximates to

$$(\theta+\alpha)_{\text{max.}} \sim \tan^{-1}\left(-\frac{c}{\surd 2v}\frac{\Omega}{p}\right) + O\left(\frac{\surd 2v}{c}\frac{p}{\Omega}\right),$$

and is an angle of the order of $-\frac{1}{2}\pi$, while the wave normal direction, θ, is $O\{(\surd 2v/c)\,p/\Omega\}$ and is almost parallel to magnetic field direction. For Čerenkov radiation in the exosphere it is no longer true to think of the ray direction lying within a cone of semi-angle 20° to the

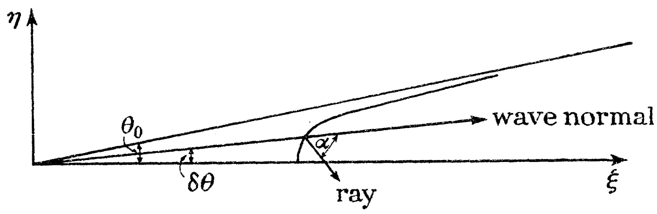

FIGURE 5. The (μ, θ) curve for region $6a$ of the (X, Y) plane when $X \gg 1$, $X/Y^2 \gg 1$.

magnetic field direction (Storey 1953) since the wave frequency may go up to a frequency just short of Ω, the gyro frequency. For example, using the typical values of v (1000 km/s) and Ω/p ($\sim 2\times 10^{-2}$) quoted in the previous section the maximum deviation of the ray from the magnetic field direction is

$$(\theta+\alpha)_{\text{max.}} \sim -86°,$$

with θ and α being 2° and $-88°$, respectively.

For waves belonging to the lower end of the Čerenkov range $\omega_1 < \omega < \omega_2$ the rays are focused along the field direction since $\theta \sim \frac{1}{2}\pi$ and $\alpha \sim -\frac{1}{2}\pi$.

(c) *Variation of frequency with time of arrival*

It is now established that a charged particle travelling along the earth's magnetic lines of force in the exosphere will generate at each point, s say, Čerenkov radiation over a frequency band $\omega \leqslant \omega_2 \sim \Omega$. Since Ω varies at the inverse cube of the distance from the earth's centre, R, ($\Omega \sim R^{-3}$) the bandwidth of the radiation increases as the particle travels towards the earth. The time of emission (Φ, say) of each range depends on the speed, v, of the particle. The time of arrival, t, at the observer of a frequency, ω, generated at a point s is the sum of the propagation 'time', τ, and the emission time, Φ,

$$t = \Phi + \tau, \tag{5·15}$$

where τ is given by

$$\tau(\omega) = \frac{1}{c}\int \mu_{g-r}(\omega)\,\mathrm{d}s, \tag{5·16}$$

the integral being taken along the ray path of which ds is an element. Substituting μ_{g-r} from equation (5·10) into equation (5·16) we have for the propagation time

$$\tau = \frac{D}{\omega}, \quad D = \frac{1}{v}\int_{R_\Phi}^{R_0} \Omega \, \mathrm{d}s. \tag{5·17}$$

R_Φ is the point at which the frequency ω ($< \omega_2(R_\Phi)$) is emitted. The time of propagation is seen to be inversely proportional to the wave frequency.

Since the group velocity U is always less than the particle velocity v frequency bands generated nearer the earth will arrive as a whole before those generated further out. The observer will therefore see a band of frequencies arriving at the same time, each frequency (ω_i, say), in the band belonging to a different Čerenkov range generated at a different point (s_i, say). The arrival spectrum is a superposition of all the spectra generated at different points s_i at emission times Φ_i. A qualitative picture of the observed spectrum is shown in figure 6 and in fact is a solid band of 'hiss', the mean frequency falling with arrival time as equation (5·17) shows.

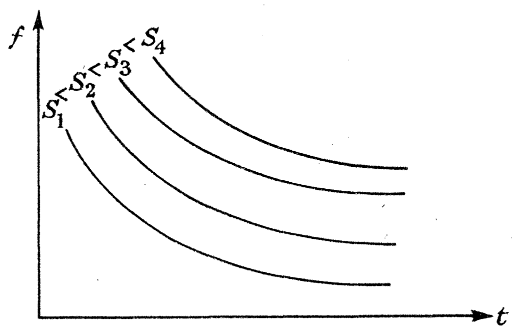

FIGURE 6. Spectrum of frequency against arrival time.

An estimate of the power radiated by a small cloud of solar corpuscular particles moving through the exosphere can be obtained from the numerical computations (see next section) of the power frequency spectrum. For solar clouds of dimensions suggested by Gallet (1959), i.e. 100 km long, with a number density appropriate to auroral protons (10^5 electrons/m^3) power intensities of the same order as low frequency noise bursts measured by Ellis (1959), i.e. 10^{-19} to 10^{-17} W m^{-2} (c/s)$^{-1}$, can be obtained. Actually this estimate should be multiplied by the Fraunhöfer diffraction factor, $\{\sin^2(\omega L/v)\}/(\omega L/v)^2$, where $2L$ is the length of the cloud, to account for the finite extension of the cloud the dimensions of which are now comparable with the wavelength of the emitted radiation. In this case the power emitted per unit frequency would be cut by a factor of the order of 10^{-6} if the frequency of the emitted radiation was around 5 kc/s. However such estimates of the power radiated are unreliable as any desired power intensity can be obtained from this mechanism by suitably juggling the length of the solar cloud, its charge density, and the path length traversed by the cloud in the exosphere. Nevertheless, it does seem likely that low-frequency hiss originates from the generation of Čerenkov radiation by clouds of charged particles streaming through the exosphere.

6. The power-frequency spectra and polar diagrams

(a) *The low-frequency approximation*

The power-frequency spectra and polar diagram of Čerenkov radiation emitted by a uniformly moving point charge, equations (3·14), respectively, assume the following simple forms over the low-frequency band when the particle speed is not relativistic, and the gyro frequency is much smaller than the plasma frequency, and the product $(c\Omega/vp)^2 \gg 1$,

$$\frac{dW_1}{d\omega} \simeq \frac{e^2\mu_0}{4\pi}\frac{\omega}{\Omega^2}(\omega_1^2-\omega^2), \quad \left(0<\omega<\omega_1 \simeq \frac{v^2p^2}{c^2\Omega}\left(1+\frac{v^2p^2}{c^2\Omega^2}\right)\right),$$

$$dW_{1,\text{ ang.}} \simeq \frac{e^2\mu_0}{8\pi^2}\left(\frac{v^2p^2}{c^2\Omega^2}\right)^4 \Omega^2 \cos\theta \sin^2\theta, \tag{6·1}$$

$$\frac{dW_2}{d\omega} \simeq \frac{e^2\mu_0}{4\pi}\frac{c^2}{v^2}\frac{\omega}{p^2}(\omega_2^2-\omega^2), \quad \left(0<\omega<\omega_2 \simeq \Omega\left(1-\frac{v^2}{c^2}\frac{p^2}{\Omega^2}\right)\right),$$

$$dW_{2,\text{ ang.}} \simeq \frac{e^2\mu_0}{8\pi^2}\left(\frac{c^2\Omega^2}{v^2p^2}\right)\Omega^2 \cos\theta \sin^2\theta. \tag{6·2}$$

The spectra exhibit maxima at $\qquad \omega_{i,\text{ max.}} = \dfrac{1}{\surd 3}\omega_i$

with the corresponding angle of maximum emission being

$$\cos\theta_i = 1/\surd 3, \quad \theta_i = 54^\circ\, 44'.$$

For particle speeds of the order 300 to 1000 km/s at distances between five to two earth radii from the earth's centre the μ_1 mode radiates over a frequency band, ω_1, varying from roughly 50 c/s for the slower speed up to 0·5 kc/s for the faster particle. Radiation at kilocycle frequencies is generated in the μ_2 mode and relative to the μ_1 mode the intensity is far greater, the ratio W_2/W_1 being $(c^2\Omega/c^2p^2)^5 \gg 1$.

An interesting feature of the radiation, brought out by equation (6·2), is that the more slowly the charge moves the greater is the emitted radiation. In this approximation the energy radiated is inversly proportional to the particle's kinetic energy. The suggestion is that slow particles are severely braked by energy loss through Čerenkov emission. This phenomenon is directly related to the existence of some branches of the refractive index, μ, reaching to infinity at certain frequencies and angles of propagation to the field direction. In these regions the effect of spatial dispersion, which is neglected here, becomes important. A more thorough investigation of the problem in regions where the refractive index becomes very large necessitates the introduction of the effect of spatial dispersion in the medium. In the appendix an estimate of the particle's energy variation with time is obtained by equating the rate of energy loss to the negative rate of working of the radiation force on the particle moving in a plasma immersed in an infinite magnetic field, and a lower limit to the speed of the charge below which our analysis is not valid is obtained by asserting that Čerenkov energy loss per unit path length must be less than the charge's kinetic energy. Here a rough estimate of this critical velocity can be made by asserting that the particle's kinetic energy must be larger than the Čerenkov energy loss in the μ_2 mode over the frequency band $0 < \omega < \omega_2$ (neglecting energy loss over other frequency bands), i.e.

$$\frac{W_2}{\frac{1}{2}mv^2} < 1, \quad W_2 \simeq \frac{e^2\mu_0}{16\pi}\frac{c^2}{v^2}\frac{\Omega^4}{p^2}, \tag{6·3}$$

which yields

$$V_{\text{crit.}} = \left(\frac{e^2\mu_0}{8\pi m}\right)^{\frac{1}{4}}\left(\frac{c\Omega^2}{p}\right)^{\frac{1}{2}}. \tag{6·4}$$

Our analysis will be valid for particle speeds greatly exceeding $V_{\text{crit.}}$. This critical velocity is of the order of 3 m/s for gyro and plasma frequencies appropriate to exosphere conditions at 5 earth radii from the earth's centre.

(*b*) *Some spectral and polar distributions of Čerenkov radiation for specific values of v at various distances from the earth's centre*

Figure 7 is a polar plot of the normalized angular distribution of radiation against wave normal. The ratio of the gyro to the plasma frequency, Ω/p, is plotted against distance from the earth's centre in figure 8 on the basis of a model of the exospheric electron density

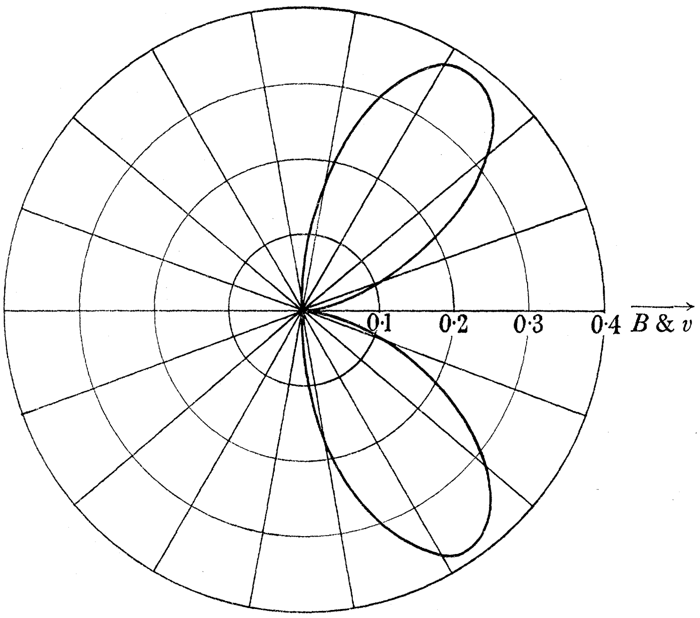

FIGURE 7. The normalized angular distribution of radiation $f(\theta)$;

$$f(\theta) \sim \cos\theta\sin^2\theta \sim \mathrm{d}W_{2,\,\text{ang.}}\left[\frac{e^2\mu_0}{8\pi^2}\left(\frac{c\Omega^2}{vp}\right)^2\right]^{-1}.$$

distribution proposed by Helliwell (1961). This curve also gives the low-frequency, non-relativistic critical velocity defined by inequality (4·13) above which there is no radiation.

The direction the ray makes with the magnetic field $(\theta+\alpha)$ is plotted as a function of the wave normal direction, θ, at two earth radii for various particle speeds in figure 9. Figure 10 illustrates the normalized angular distribution of radiation against ray direction $(\theta+\alpha)$ at various particle velocities for Ω/p appropriate to 2 earth radii, and shows that the greater the particle speed the more the radiation is directed along the particle track.

Figures 11 to 13 show the variation of the maximum emitted power,

$$dW_{2,\,\mathrm{max.}}/d\omega \sim 10^{-45}(c\Omega/vp)^2\,\tfrac{1}{2}\Omega\,\mathrm{W\,m^{-1}\,(c/s)^{-1}},$$

with distance from the centre of the earth at 100, 300 and 500 km/s, respectively. The exospheric model of electron density against height here, namely the electron number

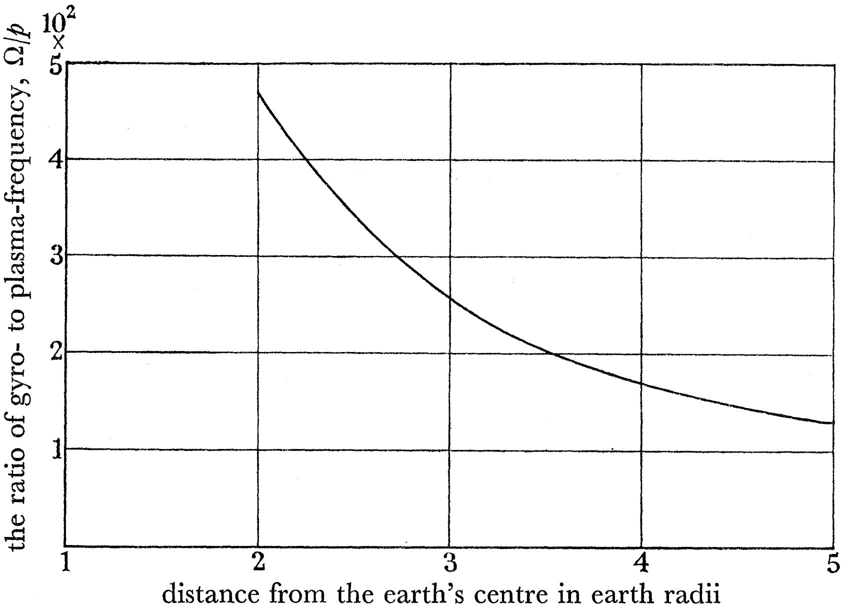

FIGURE 8. The ratio $\Omega/p(2v_e/c)$ against distance from the earth's centre in earth radii. The limiting velocity $v_e \simeq c\Omega/2p$ above which there is no radiation is essentially given by this curve.

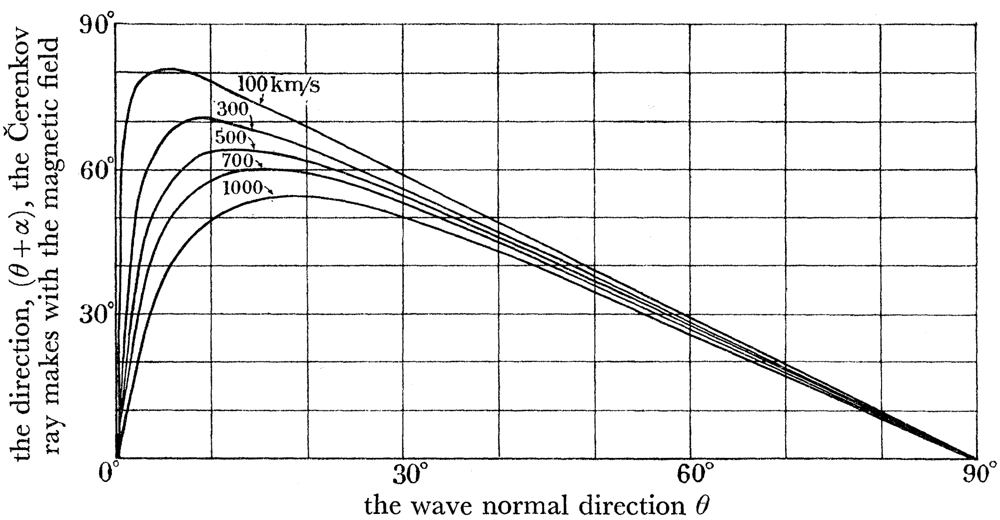

FIGURE 9. The Čerenkov ray direction $(\theta+\alpha)$ against the wave normal at 2 earth radii for various particle speeds.

density (proportional to the square of the plasma frequency) is proportional to the strength of the earth's magnetic field, yields that the power emitted per unit frequency range is proportional to the square of the gyro frequency and therefore increases as the particle approaches the earth.

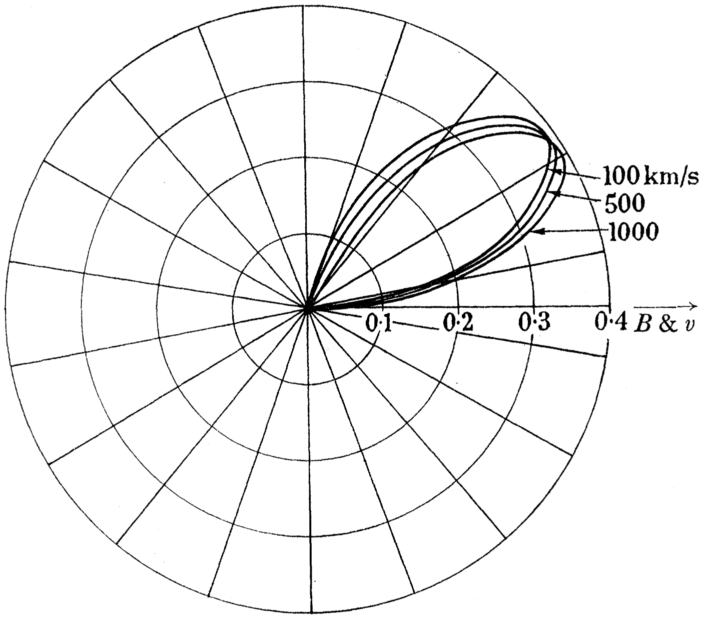

FIGURE 10. Polar plot of normalized energy distribution against ray direction $(\theta+\alpha)$ for various particle speeds at 2 earth radii.

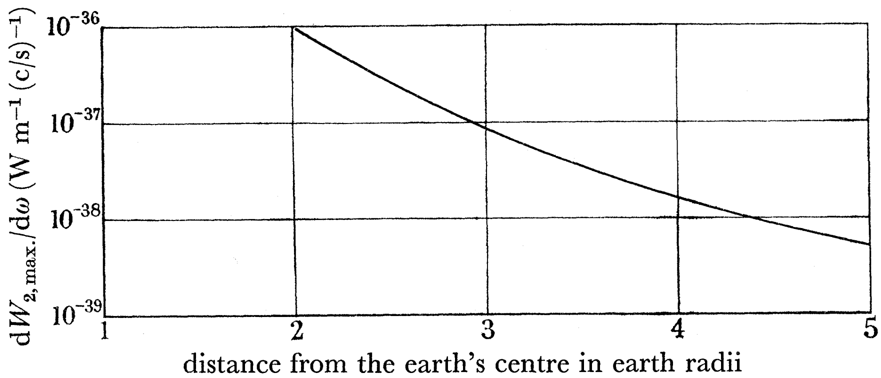

FIGURE 11. $dW_{2,\max.}/d\omega$ against distance from the earth's centre for a particle speed of 100 km/s.
$dW_{2,\max.}/d\omega \simeq 10^{-45}(c\Omega/vp)^2\ \Omega/2$

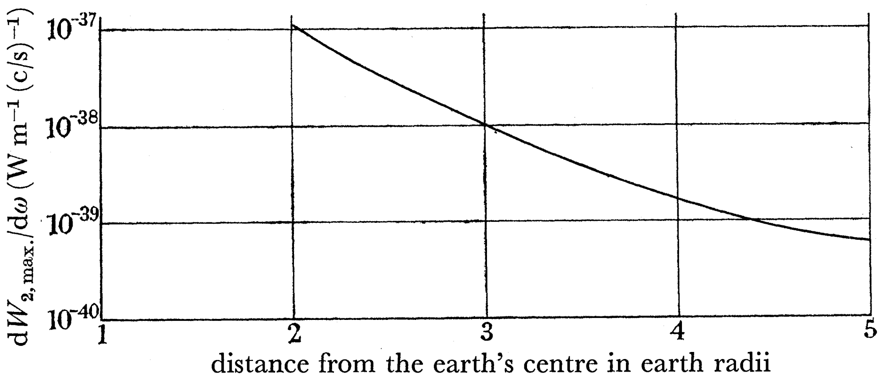

FIGURE 12. $dW_{2,\max.}/d\omega$ against distance from the earth's centre for a particle speed of 300 km/s.

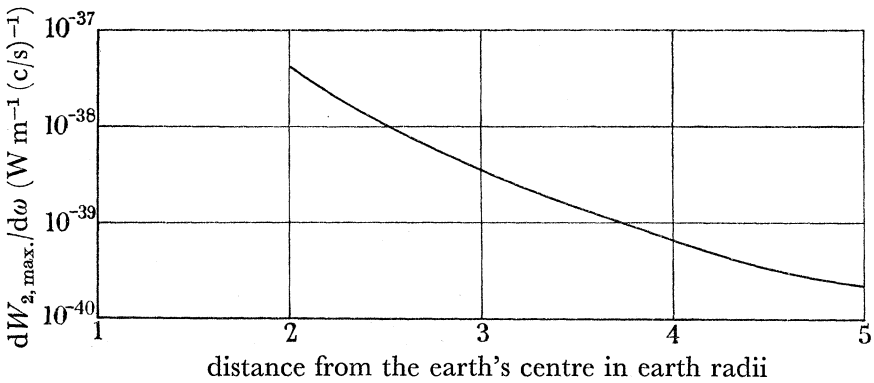

FIGURE 13. $dW_{2,\max.}/d\omega$ against distance from the earth's centre for a particle speed of 500 km/s.

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APPENDIX. ČERENKOV RADIATION IN A COLD PLASMA IMMERSED IN AN INFINITE MAGNETOSTATIC FIELD

Radiation from a point charge in uniform motion in a cold plasma immersed in an infinite magnetostatic field is considered. The magneto-ionic tensor becomes diagonal with the components transverse to the magnetic field direction equal, the medium therefore assumes the properties of a uniaxial crystal with the two characteristic waves described by the refractive indices μ_1^2 (obtained from equation (2·10) by letting $\epsilon \to \infty$) given by

$$\left.\begin{aligned} \mu_1^2 &= 1, \\ \mu_2^2 &= 1 + X(l^2 - 1). \end{aligned}\right\} \tag{A 1}$$

One wave, the ordinary wave, propagates as if the medium were free space and therefore Čerenkov radiation in this mode is not possible. The extraordinary wave is a function of frequency and wave normal direction and the refractive index in terms of θ and X is

$$\mu_2^2 = \frac{X-1}{X\cos^2\theta - 1}. \tag{A 2}$$

The (μ, θ) curves have distinctly different characteristics for $X \gtrless 1$. If $X < 1$ the curves are ellipses with semi-major and semi-minor axes (i.e. along and perpendicular to the field direction, respectively) 1 and $(1-X)^{\frac{1}{2}}$, respectively. Therefore no Čerenkov radiation is possible if the wave frequency is greater than the plasma frequency since the speed of phase propagation is greater or equal to the vacuum velocity of light. When $X > 1$ the (μ, θ) curves are hyperboles reaching asymptotically to infinity in the direction $\cos\theta_0 = 1/X^{\frac{1}{2}}$. Čerenkov radiation is therefore generated over the frequency band $0 < \omega < p$.

If the charge moves along the direction of the infinite magnetic field with speed v the field generated and the spectral and polar distributions of radiation are given by equations (3·7), (3·8), (3·9), (3·14) with $\gamma = 0$, $\alpha = 1$. The formulae for the power spectrum, total energy loss and angular distribution of the radiation per unit path length are

$$\frac{dW}{d\omega} = \frac{e^2\mu_0}{4\pi}\left(\frac{c^2}{v^2} - 1\right)\omega \quad (0 < \omega < p), \tag{A 3}$$

$$W = \frac{e^2\mu_0}{8\pi} p^2 \left(\frac{c^2}{v^2} - 1\right), \tag{A 4}$$

$$\mathrm{d}W_{\text{ang.}} = \frac{e^2\mu_0}{8\pi^2} \frac{c^2}{v^2} p^2 \frac{(1 - v^2/c^2)^2 \cos\theta}{(1 - (v\cos\theta/c)^2)^2}. \tag{A 5}$$

The oddity in the above expressions (as in equation (6·2)) is that slow particles radiate strongly while fast, relativistic particles suffer only a slight energy loss by Čerenkov radiation. Slow particles will therefore be stopped quickly by the radiation reaction force. An estimate of the energy variation with time for a particle losing energy by Čerenkov emission may be obtained by equating the rate of energy loss to the negative rate of working of the reaction force on the particle. Accordingly we have

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -Wv = -\frac{e^2\mu_0}{8\pi} p^2 \left(\frac{c^2}{v^2} - 1\right) v, \tag{A 6}$$

where ξ is the particle energy $mc^2(1 - v^2/c^2)^{-\frac{1}{2}}$. Putting $\xi = mc^2\eta$, equation (A 6) may be written

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{e^2\mu_0}{8\pi} \frac{p^2}{mc} \frac{1}{\eta(\eta^2 - 1)^{\frac{1}{2}}}, \tag{A 7}$$

whence

$$t = \frac{8\pi mc}{3e^2\mu_0} \{(\eta_0^2 - 1)^{\frac{3}{2}} - (\eta^2 - 1)^{\frac{3}{2}}\}, \tag{A 8}$$

where $mc^2\eta_0$ is the intial energy of the particle at the initial time $t = 0$. The time, t_{stop}, for the particle to lose all its energy through Čerenkov radiation is given by equation (A 8) with $\eta^2 = 1$, which becomes

$$t_{\text{stop}} \sim 7 \times 10^{22} (\eta_0^2 - 1)^{\frac{3}{2}} / p^2. \tag{A 9}$$

(*a*) *Non-relativistic particles*

$$t_{\text{stop}} \sim 7 \times 10^{22} (v_0/c)^3 / p^2 \to 0 \quad \text{as} \quad v_0 \to 0.$$

(*b*) *Relativistic particles*

$$t_{\text{stop}} \sim 2{\cdot}5 \times 10^{22} (v_0/c)^3 / p^2 (1 - (v_0/c)^2) \to \infty \quad \text{as} \quad v_0 \to c.$$

A critical velocity below which our analysis is invalid (since the reaction force is neglected) can be obtained by asserting that the particle's kinetic energy must exceed the Čerenkov energy loss per unit length of path. For non-relativistic particles this critical velocity is found to be

$$v_{\text{crit.}} \sim 4p^{\frac{1}{2}} \text{ m/s}. \tag{A 10}$$

Using a typical plasma frequency for the exosphere, $p \sim 10^6$ c/s, we see that our analysis is valid for particle speeds greatly exceeding 4 km/s.

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